INTEGRALI INDEFINITI e DEFINITI

Esercizi proposti

1. Calcolare i seguenti integrali indefiniti "immediati":

$(a) \int \frac{\log^3 x}{x}\, dx \qquad (b) \int \frac{dx}{x \log^3 x} \qquad (c) \int x^2 e^{x^3}\, dx$

$(d) \int \frac{\text{arctg}^4 x}{1+x^2}\, dx \qquad (e) \int \frac{x}{\sqrt{(1-x^2)^3}}\, dx \qquad (f) \int \frac{1+\cos x}{x+\sin x}\, dx$

$(g) \int \frac{x^3}{1+x^8}\, dx \qquad (h) \int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\, dx \qquad (i) \int \frac{\sin 2x}{1+\sin^2 x}\, dx.$

2. Utilizzando la proprietà di linearità, calcolare i seguenti integrali:

$(a) \int (4x^4 + 3x^2 + 5x)\, dx \qquad (b) \int \frac{x^3+x+1}{x^2+1}\, dx \qquad (c) \int (1+2x^3)^2\, dx$

$(d) \int (1+\cos x)^2\, dx \qquad (e) \int \text{ctg}^2 x\, dx \qquad (f) \int \cos^3 x\, dx.$

3. Calcolare i seguenti integrali con la tecnica di integrazione per parti:

$(a) \int x^3 \text{sh} x\, dx \qquad (b) \int x^3 \sin(x^2)\, dx \qquad (c) \int x^4 \cos(2x)\, dx$

$(d) \int e^{2x} \sin(3x)\, dx \qquad (e) \int e^{-3x} \cos(2x)\, dx \qquad (f) \int \arcsin x\, dx$

$(g) \int x^3 \log x\, dx \qquad (h) \int x^5 e^{-x^3}\, dx \qquad (i) \int \frac{\log x}{\sqrt[4]{x}}\, dx$

$(j) \int \log^2 x\, dx \qquad (k) \int x \sin^2 x\, dx \qquad (l) \int \log(\sqrt{x+1} + \sqrt{x-1})\, dx.$

4. Calcolare i seguenti integrali di funzioni razionali:

$(a) \int \frac{x^2-2x-1}{x^2-4x+4}\, dx \qquad (b) \int \frac{x^2-10x+10}{x^3+2x^2+5x}\, dx \qquad (c) \int \frac{3x^2-x}{(x+1)^2(x+2)}\, dx$

$(d) \int \frac{dx}{x^4-1} \qquad (e) \int \frac{x^3-2}{x^2(x^2+1)}\, dx \qquad (f) \int \frac{x^3}{x^2+7x+12}\, dx$

5. Calcolare i seguenti integrali effettuando le opportune sostituzioni:

$(a) \int \frac{dx}{x(2+\log^2 x)} \qquad (b) \int \frac{x^3}{\sqrt{1-x^2}}\, dx \qquad (c) \int \frac{x^5}{\sqrt{x^3-1}}\, dx$

$(d) \int \sqrt{e^x - 1}\, dx \qquad (e) \int \frac{1}{\sqrt{(1-x^2)^3}}\, dx \qquad (f) \int \frac{1}{x^2\sqrt{1+x^2}}\, dx$

$(g) \int \frac{e^{3x}+2e^{2x}+3e^x}{e^x+1}\, dx \qquad (h) \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} \qquad (i) \int \frac{1-3x}{\sqrt{x}-2}\, dx$

$(j) \int \frac{dx}{2\sin x + \cos x - 1} \qquad (k) \int \frac{\sin 2x}{6\sin x - \cos 2x + 5}\, dx \qquad (l) \int \text{ctg}^5 x\, dx$

$(m) \int \frac{\text{tg}^3 x + \text{tg} x}{\text{tg} x + 4}\, dx \qquad (n) \int \frac{\text{tg} x}{\sin^2 x + 1}\, dx \qquad (o) \int x \log(1-2x-3x^2)\, dx.$

6. Calcolare i seguenti integrali definiti:

   a) $\displaystyle\int_6^8 \frac{x^2-5x+4}{x-5}\,dx$

   b) $\displaystyle\int_0^1 x^2 \arctan x\,dx$

   c) $\displaystyle\int_0^1 \frac{2x^2+x+4}{(x^2+1)(x+2)}\,dx$

   d) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2|\cos x|\,dx.$

7. Calcolare l'area A delle seguenti regioni del piano (O, x, y):

   a) regione compresa tra il grafico della funzione $f(x) = \dfrac{9x}{(x+1)(x-2)}$ e l'asse delle x, per $x \in [0,1]$;

   b) regione compresa tra il grafico della funzione $f(x) = x \sin 2x$ e l'asse delle x, per $x \in [0,\pi]$;

   c) regione compresa tra il grafico della funzione $f(x) = \dfrac{1}{x^2\sqrt{x^2-1}}$ e l'asse delle x, per $x \in [\sqrt{2}, 2]$.

8. Sia $f(x) = \sin^3 x \cos^2 x$. Si calcoli la media integrale μ di f sull'intervallo $\left[0, \dfrac{\pi}{3}\right]$; si dica se esiste un punto $c \in \left[0, \dfrac{\pi}{3}\right]$ per cui $f(c) = \mu$.

9. Sia $g(x) = (1+x^2)e^{-|x+1|}$. Si calcoli la primitiva G di g in $\mathbb{R}$ tale che $\lim_{x\to+\infty} G(x) = 3$.

10. Data la funzione
$$f(x) = \begin{cases} x^3 \sin(\pi x^2) & x \leq 1 \\ x^2 - 8x + 16 & x > 1, \end{cases}$$
determinare la primitiva generalizzata di f che si annulla in $x_0 = 0$.

11. a) Si calcoli $I_n = \int_1^2 \frac{nx}{(x^2+\frac{1}{n})^n}\,dx$, $\forall n \in \mathbb{N}$, $n > 0$.
    b) Si calcoli $\lim_{n\to+\infty} I_n$.

12. Si calcoli la primitiva che si annulla in $x_0 = 0$ della seguente funzione definita su $(-\infty, 3)$
$$f(x) = \frac{x+2}{(|x|+3)(x-3)}.$$

13. Data la funzione
$$f(x) = \begin{cases} x|6x-2|+1 & x \leq 0 \\ (x+1)e^{x/2} & x > 0, \end{cases}$$

   a) determinare la primitiva F di f tale che $F(-1) = 0$;

   b) calcolare $\int_{-1}^2 f(t)\,dt$;

   c) calcolare $\int_{-1/3}^{2/3} f(3t)\,dt$.

Risposte agli esercizi

1.

(a) $\int \frac{\log^3 x}{x}\,dx = \frac{\log^4 x}{4} + c$ (b) $\int \frac{dx}{x\log^3 x} = -\frac{1}{2\log^2 x} + c$ (c) $\int x^2 e^{x^3}\,dx = \frac{e^{x^3}}{3} + c$

(d) $\int \frac{\text{arctg}^4 x}{1+x^2}\,dx = \frac{\text{arctg}^5 x}{5} + c$ (e) $\int \frac{x\,dx}{\sqrt{(1-x^2)^3}} = \frac{1}{\sqrt{1-x^2}} + c$ (f) $\int \frac{1+\cos x}{x+\sin x}\,dx = \log|x+\sin x| + c$

(g) $\int \frac{x^3\,dx}{1+x^8} = \frac{\text{arctg}(x^4)}{4} + c$ (h) $\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx = \frac{\arcsin^3 x}{3} + c$ (i) $\int \frac{\sin 2x\,dx}{1+\sin^2 x} = \log(1+\sin^2 x) + c.$

2.

(a) $\int (4x^4 + 3x^2 + 5x)\,dx = \frac{4}{5}x^5 + x^3 + \frac{5}{2}x^2 + c$ (b) $\int \frac{x^3+x+1}{x^2+1}\,dx = \frac{x^2}{2} + \text{arctg}x + c$

(c) $\int (1+2x^3)^2\,dx = \frac{4}{7}x^7 + x^4 + x + c$ (d) $\int (1+\cos x)^2\,dx = \frac{3}{2}x + 2\sin x + \frac{\sin x\cos x}{2} + c$

(e) $\int \text{ctg}^2 x\,dx = -x - \text{ctg}x + c$ (f) $\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3} + c.$

3.

(a) $\int x^3 \text{sh}x\,dx = (x^3+6x)\text{ch}x - 3(x^2+2)\text{sh}x + c$ (b) $\int x^3 \sin(x^2)\,dx = \frac{-x^2\cos(x^2) + \sin(x^2)}{2} + c$

(c) $\int x^4\cos(2x)\,dx = \frac{1}{2}\sin(2x)(x^4 - 3x^2 + \frac{3}{2}) + \cos(2x)(x^3 - \frac{3}{2}x) + c$

(d) $\int e^{2x}\sin(3x)\,dx = \frac{2}{13}e^{2x}[\sin(3x) - \frac{3}{2}\cos(3x)]$ (e) $\int e^{-3x}\cos(2x)\,dx = -\frac{3}{13}e^{-3x}[\cos(2x) - \frac{2}{3}\sin(2x)]$

(f) $\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + c$ (g) $\int x^3\log x\,dx = \frac{x^4\log x}{4} - \frac{x^4}{16} + c$

(h) $\int x^5 e^{-x^3}\,dx = \frac{e^{-x^3}(-x^3-1)}{3} + c$ (i) $\int \frac{\log x}{\sqrt[4]{x}}\,dx = \frac{4}{3}x^{3/4}\log x - \frac{16}{9}x^{3/4} + c$

(j) $\int \log^2 x\,dx = x(\log^2 x - 2\log x + 2) + c$ (k) $\int x\sin^2 x\,dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x) + c$

(l) $\int \log(\sqrt{x+1} + \sqrt{x-1})\,dx = x\log(\sqrt{x+1} + \sqrt{x-1}) - \frac{\sqrt{x^2-1}}{2} + c.$

4.

(a) $\int \frac{x^2-2x-1}{x^2-4x+4}\,dx = x + 2\log|x-2| + \frac{1}{x-2} + c$

(b) $\int \frac{x^2-10x+10}{x^3+2x^2+5x}\,dx = 2\log|x| - \frac{1}{2}\log(x^2+2x+5) - \frac{13}{2}\text{arctg}\frac{x+1}{2} + c$

(c) $\int \frac{3x^2-x}{(x+1)^2(x+2)}\,dx = 14\log|x+2| - 11\log|x+1| - \frac{4}{x+1} + c$

(d) $\int \frac{dx}{x^4-1} = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\text{arctg}x + c$

(e) $\int \frac{x^3-2}{x^2(x^2+1)}\,dx = \frac{1}{2}\log(x^2+1) + \frac{2}{x} + 2\text{arctg}x + c$

(f) $\int \frac{x^3}{x^2+7x+12}\,dx = \frac{x^2}{2} - 7x + 64\log|x+4| - 27\log|x+3| + c.$

5.

$$(a) \int \frac{dx}{x(2+\log^2 x)} = \frac{1}{\sqrt{2}}\text{arctg}\left(\frac{\log x}{\sqrt{2}}\right) + c \qquad (b) \int \frac{x^3}{\sqrt{1-x^2}}\ dx = -\frac{1}{3}(x^2+2)\sqrt{1-x^2} + c$$

$$(c) \int \frac{x^5}{\sqrt{x^3-1}}\ dx = \frac{2}{9}\sqrt{(x^3-1)^3} + \frac{2}{3}\sqrt{x^3-1} + c \quad (d) \int \sqrt{e^x-1}\ dx = 2\sqrt{e^x-1} - 2\text{arctg}\sqrt{e^x-1} + c$$

$$(e) \int \frac{1}{\sqrt{(1-x^2)^3}}\ dx = \frac{x}{\sqrt{1-x^2}} + c \qquad (f) \int \frac{1}{x^2\sqrt{1+x^2}}\ dx = -\frac{\sqrt{1+x^2}}{x} + c$$

$$(g) \int \frac{e^{3x}+2e^{2x}+3e^x}{e^x+1}\ dx = \frac{1}{2}e^{2x} + e^x + 2\log(e^x+1) + c$$

$$(h) \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log(1+\sqrt[6]{x}) + c$$

$$(i) \int \frac{1-3x}{\sqrt{x}-2}\ dx = -22\sqrt{x} - 2x\sqrt{x} - 44\log(\sqrt{x}-2) - 6x + c$$

$$(j) \int \frac{dx}{2\sin x+\cos x-1} = \frac{1}{2}\log\left|\text{tg}\frac{x}{2}\right| - \frac{1}{2}\log\left|\text{tg}\frac{x}{2}-2\right| + c$$

$$(k) \int \frac{\sin 2x}{6\sin x-\cos 2x+5}\ dx = 2\log|\sin x+2| - \log|\sin x+1| + c$$

$$(l) \int \text{ctg}^5 x\ dx = -\frac{1}{4\tan^4 x} + \frac{1}{2\tan^2 x} + \log|\sin x| + c$$

$$(m) \int \frac{\text{tg}^3 x+\text{tg}x}{\text{tg}x+4}\ dx = \text{tg}x - 4\log|\text{tg}x+4| + c$$

$$(n) \int \frac{\text{tg}x}{\sin^2 x+1}\ dx = \frac{1}{4}\log(1+2\text{tg}^2 x) + c$$

$$(o) \int x\log(1-2x-3x^2)\ dx = \frac{1}{2}x^2\log(1-2x-3x^2) - \frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{18}\log\left|x-\frac{1}{3}\right| - \frac{1}{2}\log|x+1| + c.$$

6. a) $\displaystyle\int_6^8 \frac{x^2-5x+4}{x-5}\ dx = 14 + 4\log 3$

b) $\displaystyle\int_0^1 x^2\arctan x\ dx = \frac{\pi}{12} + \frac{1}{6}(\log 2 - 1)$

c) $\displaystyle\int_0^1 \frac{2x^2+x+4}{(x^2+1)(x+2)}\ dx = \frac{\pi}{4} + 2\log\frac{3}{2}$

d) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2|\cos x|\ dx = 3\pi^2 + 4\pi - 4.$

7.

$$(a)\ \ A = 3\log 2 \qquad (b)\ \ A = \pi \qquad (c)\ \ A = \frac{\sqrt{3}-\sqrt{2}}{2}.$$

8. $\mu = \dfrac{47}{160\pi}$; esiste un punto $c \in \left[0, \frac{\pi}{3}\right]$ per cui $f(c) = \mu$, perché $f(x)$ è continua nell'intervallo $\left[0, \frac{\pi}{3}\right]$.

9.
$$G(x) = \begin{cases} -e^{-(x+1)}\left(x^2+2x+3\right)+3, & \text{se } x \geq -1 \\ e^{x+1}\left(x^2-2x+3\right)-5, & \text{se } x < -1. \end{cases}$$

10.
$$F(x) = \begin{cases} \dfrac{1}{2\pi^2}\left[-\pi x^2\cos(\pi x^2)+\sin(\pi x^2)\right], & \text{se } x \leq 1 \\ \dfrac{1}{3}x^3-4x^2+16x+\dfrac{1}{2\pi}-\dfrac{37}{3}, & \text{se } x > 1. \end{cases}$$

11. a)
$$I_n = \begin{cases} \dfrac{n}{2-2n}\left[\left(4+\dfrac{1}{n}\right)^{1-n}-\left(1+\dfrac{1}{n}\right)^{1-n}\right], & \text{se } n \neq 1 \\ \dfrac{1}{2}\log\dfrac{5}{2}, & \text{se } n = 1. \end{cases}$$

b) $\lim_{n\to+\infty} I_n = \dfrac{1}{2e}$.

12.
$$F(x) = \begin{cases} \dfrac{1}{6}\log(x+3)+\dfrac{5}{6}\log(3-x)-\log 3, & \text{se } 0 \leq x < 3 \\ \dfrac{5}{x-3}-\log(3-x)+\dfrac{5}{3}+\log 3, & \text{se } x < 0. \end{cases}$$

13. a) $F(x) = \begin{cases} -2x^3+x^2+x-2, & \text{se } x \leq 0 \\ 2(x-1)e^{x/2}, & \text{se } x > 0. \end{cases}$

b) $\displaystyle\int_{-1}^{2} f(t)\,dt = 2e$

c) $\displaystyle\int_{-1/3}^{2/3} f(3t)\,dt = \frac{2}{3}e.$